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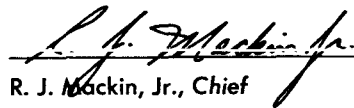
**JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA**

April 15, 1964

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***A Simple Proof of the Identity of Landau's
and van Kampen's Solutions of the
Linearized Vlasov Equation***

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ABSTRACT

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In a generalized analysis, Case has shown that the electrostatic potentials obtained in Landau's and van Kampen's solutions to the collisionless Boltzmann equation are equivalent. The purpose of this Report is to furnish a simple explicit proof of the complete identity of Landau's and van Kampen's solutions; the analysis, however, follows a course different from that of Case.

Author

I. INTRODUCTION

Two distinct methods have been used in the self-consistent-field theory of plasma oscillations: the Laplace transform technique, first used by Landau (Ref. 1), and the normal-mode expansion, first developed by van Kampen (Ref. 2). These authors solved the initial-value problem of the linearized Vlasov equation and obtained solutions which appear rather different in form. In 1959, an elegant discussion on the same problem was given by Case (Ref. 3), who generalized on van Kampen's work, treating the case in which a discrete set of eigenvalues may exist, in addition to the continuum set of

eigenvalues considered by van Kampen. Also, Case showed equivalence of the electrostatic potentials obtained by the two methods.

In the present analysis, which follows a course different from that of Case, an attempt is made to provide a simple explicit proof of the complete identity of the Landau and van Kampen solutions. Comparison of the two solutions, together with proof of the equivalence, may enable us to achieve a better understanding of the problem.

II. STATEMENT OF THE MATHEMATICAL PROBLEM

If we denote the perturbed electron distribution function by $f(\mathbf{r}, \mathbf{v}, t)$ and define its Fourier transform $f(\mathbf{k}, \mathbf{v}, t)$ by

$$f(\mathbf{k}, \mathbf{v}, t) = \int d\mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} f(\mathbf{r}, \mathbf{v}, t) \quad (1)$$

then the linearized Vlasov equation takes the form

$$\frac{\partial f}{\partial t} + i\mathbf{k} \cdot \mathbf{v} f = -iD(\mathbf{v})k \int d\mathbf{v} f(\mathbf{k}, \mathbf{v}, t) \quad (2)$$

where

$$D(\mathbf{v}) = -\frac{4\pi n e^2}{m k^3} \mathbf{k} \cdot \frac{\partial f^0}{\partial \mathbf{v}}$$

Here, $f^0(\mathbf{v})$ is the electron Maxwellian distribution function, e is the electron charge, and m is the electron mass. So far, we have assumed that the ions are uniformly distributed and form a neutralizing background; therefore, the ions do not appear in the theory. The same assumption was used in Refs. 1 and 2 and is certainly justifiable because of the heavy ionic mass.

To simplify the problem further, we may introduce a reduced distribution function $\bar{f}(\mathbf{k}, u, t)$, which is defined as

$$\bar{f}(\mathbf{k}, u, t) = \int d^3v \delta\left(u - \frac{\mathbf{v} \cdot \mathbf{k}}{k}\right) f(\mathbf{k}, \mathbf{v}, t) \quad (3)$$

Thus, Eq. 2 can be reduced to the form

$$\frac{\partial \bar{f}}{\partial t} + iku\bar{f} = -ik\bar{D}(u) \int_{-\infty}^{+\infty} du \bar{f}(\mathbf{k}, u, t) \quad (4)$$

where

$$\bar{D}(u) = -\frac{4\pi n e^2}{m k^2} \frac{\partial \bar{f}^0}{\partial u}$$

The Landau solution of Eq. 4, in terms of the Laplace transform, has the form

$$\begin{aligned} \bar{f}_1(\mathbf{k}, u, \omega) &= \frac{1}{i(\omega + ku)} \\ &\times \left[\bar{f}_1(\mathbf{k}, u, 0) - \frac{k\bar{D}(u)}{\epsilon\left(\frac{\omega}{k}\right)} \int_c du \frac{\bar{f}_1(\mathbf{k}, u, 0)}{(\omega + ku)} \right] \end{aligned} \quad (5)$$

where

$$\bar{f}_1(\mathbf{k}, u, \omega) = \int_0^{+\infty} dt e^{-i\omega t} \bar{f}_1(\mathbf{k}, u, t) \quad (6)$$

$$\epsilon\left(\frac{\omega}{k}\right) = 1 + \int_c du \frac{\bar{D}(u)}{u + \frac{\omega}{k}} \quad (7)$$

and c denotes the Landau contour, which may be realized as a geometric representation of an analytic continuation of the integral from the lower half to the upper half of the complex ω plane. A detailed discussion of this analysis may be found in Ref. 1 or 4 and will therefore be omitted here.

On the other hand, van Kampen's solution, which we call $\bar{f}_2(\mathbf{k}, u, t)$, takes the form

$$\begin{aligned} \bar{f}_2(\mathbf{k}, u, t) &= \int_{-\infty}^{+\infty} d\alpha e^{-ik\alpha} [\delta_{(+)}(u - \alpha) \epsilon_{(+)}(u) + \delta_{(-)}(u - \alpha) \epsilon_{(-)}(u)] \\ &\times \int_{-\infty}^{+\infty} du' \left[\frac{\delta_{(+)}(\alpha - u')}{\epsilon_{(+)}(\alpha)} + \frac{\delta_{(-)}(\alpha - u')}{\epsilon_{(-)}(\alpha)} \right] \bar{f}(\mathbf{k}, u', 0) \end{aligned} \quad (8)$$

where

$$\delta_{(\pm)}(x) = \frac{1}{2} \delta(x) \pm \frac{i}{2\pi} P \frac{1}{x} \quad (9)$$

$$\epsilon_{(\pm)}(\alpha) = 1 \pm 2\pi i \bar{D}_{(\pm)}(\alpha) \quad (10)$$

and

We now wish to prove that the inverse Laplace transform of $\bar{f}_1(k, u, \omega)$ (the Landau solution) is identical to $\bar{f}_2(k, u, t)$ (the van Kampen solution); i.e.,

$$\bar{D}_{(\pm)}(\alpha) = \int_{-\infty}^{+\infty} du \delta_{(\pm)}(\alpha - u) \bar{D}(u) \quad (11)$$

$$\bar{f}_2(k, u, t) = \frac{1}{2\pi} \int_L d\omega e^{i\omega t} \bar{f}_1(k, u, \omega) \quad (12)$$

Here, $\delta(x)$ is the Dirac delta function, P denotes the principal value, and $\alpha = \omega/k$ (the phase velocity).

where L denotes the path of integration which is parallel to the real axis and lies in the lower half of the complex ω plane, below all the singularities of $\bar{f}_1(k, u, \omega)$.

III. PROOF

Making use of the definitions in Eqs. 9, 10, and 11, we may rewrite Eq. 8 in the following form:

$$\begin{aligned} \bar{f}_2(k, u, t) &= \int_{-\infty}^{+\infty} d\alpha e^{-i\alpha t} \left[\left(1 + P \int_{-\infty}^{+\infty} \frac{\bar{D}(u') du'}{u' - \alpha} \right) \delta(u - \alpha) - P \frac{\bar{D}(u)}{u - \alpha} \right] \\ &\quad \times \int_{-\infty}^{+\infty} du' \left[\frac{\delta_{(+)}(\alpha - u')}{\epsilon_{(+)}(\alpha)} + \frac{\delta_{(-)}(\alpha - u')}{\epsilon_{(-)}(\alpha)} \right] \bar{f}_2(k, u', 0) \\ &= \left(1 + P \int_{-\infty}^{+\infty} \frac{\bar{D}(u') du'}{u' - u} \right) e^{-i\alpha t} \left[\frac{\bar{f}_{2(+)}(k, u, 0)}{\epsilon_{(+)}(u)} + \frac{\bar{f}_{2(-)}(k, u, 0)}{\epsilon_{(-)}(u)} \right] \\ &\quad - P \int_{-\infty}^{+\infty} d\alpha \frac{\bar{D}(u)}{u - \alpha} e^{-i\alpha t} \left[\frac{\bar{f}_{2(+)}(k, \alpha, 0)}{\epsilon_{(+)}(\alpha)} + \frac{\bar{f}_{2(-)}(k, \alpha, 0)}{\epsilon_{(-)}(\alpha)} \right] \quad (13) \end{aligned}$$

Evidently,

$$\begin{aligned} \bar{f}_2(k, u, 0) &= \left(1 + P \int_{-\infty}^{+\infty} du' \frac{\bar{D}(u')}{u' - u} \right) \left[\frac{\bar{f}_{2(+)}(k, u, 0)}{\epsilon_{(+)}(u)} + \frac{\bar{f}_{2(-)}(k, u, 0)}{\epsilon_{(-)}(u)} \right] \\ &\quad - P \int_{-\infty}^{+\infty} d\alpha \frac{\bar{D}(u)}{u - \alpha} \left[\frac{\bar{f}_{2(+)}(k, \alpha, 0)}{\epsilon_{(+)}(\alpha)} + \frac{\bar{f}_{2(-)}(k, \alpha, 0)}{\epsilon_{(-)}(\alpha)} \right] \quad (14) \end{aligned}$$

Multiplying Eq. 14 by e^{-ikut} and subtracting this result from Eq. 13, one finds that

$$\begin{aligned} \bar{f}_2(k, u, t) &= \bar{f}_2(k, u, 0) e^{-ikut} \\ &+ \int_{-\infty}^{+\infty} d\alpha \frac{\bar{D}(u)}{u - \alpha} (e^{-ikut} - e^{-ikat}) \\ &\times \left[\frac{\bar{f}_{2(+)}(k, \alpha, 0)}{\varepsilon_{(+)}(\alpha)} + \frac{\bar{f}_{2(-)}(k, \alpha, 0)}{\varepsilon_{(-)}(\alpha)} \right] \end{aligned} \quad (15)$$

In Eq. 15, the symbol P (which denotes the principal value in front of the integral) is dropped, since the numerator of that integral vanishes automatically at $\alpha = u$.

Again, since the function $\frac{\bar{f}_{2(-)}(k, \alpha, 0)}{\varepsilon_{(-)}(\alpha)}$ has an analytic continuation without singularity in the lower half of the complex α plane, the integral

$$\int_{-\infty}^{+\infty} d\alpha \frac{1}{\alpha - u} (e^{-ikut} - e^{-ikat}) \left[\frac{\bar{f}_{2(-)}(k, \alpha, 0)}{\varepsilon_{(-)}(\alpha)} \right]$$

vanishes. Thus, Eq. 15 reduces to

$$\begin{aligned} \bar{f}_2(k, u, t) &= \bar{f}_2(k, u, 0) e^{-ikut} \\ &+ \int_{-\infty}^{+\infty} d\alpha \frac{\bar{D}(u)}{u - \alpha} (e^{-ikut} - e^{-ikat}) \frac{\bar{f}_{2(+)}(k, \alpha, 0)}{\varepsilon_{(+)}(\alpha)} \end{aligned} \quad (16)$$

As far as van Kampen's solution is concerned, this is the form that we prefer for the present proof.

Now, let us return to the Landau solution and study its inverse Laplace transform; i.e.,

$$\begin{aligned} \bar{f}_1(k, u, t) &= \frac{1}{2\pi} \int_{-\infty-i\gamma}^{+\infty-i\gamma} d\tilde{\omega} \frac{e^{i\tilde{\omega}t}}{i(\tilde{\omega} + ku)} \\ &\times \left[\bar{f}_1(k, u, 0) - \frac{k\bar{D}(u)}{\varepsilon\left(\frac{\tilde{\omega}}{k}\right)} \int_c du \frac{\bar{f}_1(k, u, 0)}{\tilde{\omega} + ku} \right] \end{aligned} \quad (17)$$

It is well known that, in the case of a Maxwellian $f^0(u)$, the function $\varepsilon(\omega/k)$ as defined by Eq. 7 is analytic for $\text{Im}(\tilde{\omega}) < 0$; hence, we may deform the path of integration to the real axis: i.e., we take the limit $\gamma \rightarrow 0_{(+)}$. Then Eq. 17 may be re-expressed as

$$\begin{aligned} \bar{f}_1(k, u, t) &= \lim_{\gamma \rightarrow 0_{(+)}} \frac{1}{2\pi i} \int_{-\infty-i\gamma}^{+\infty-i\gamma} d\tilde{\alpha} \frac{e^{i\tilde{\alpha}t}}{(\tilde{\alpha} + u)} \\ &\times \left[\bar{f}_1(k, u, 0) - 2\pi i \frac{\bar{D}(u)}{\varepsilon_{(-)}(\tilde{\alpha})} \bar{f}_{1(-)}(k, \tilde{\alpha}, 0) \right] \end{aligned} \quad (18)$$

where, for convenience, we have introduced the quantity $\tilde{\alpha} = \tilde{\omega}/k$ and the notations

$$2\pi i \bar{f}_{1(-)}(k, \tilde{\alpha}, 0) = \lim_{\sigma \rightarrow 0_{+}} \int_{-\infty}^{+\infty} du \frac{\bar{f}(k, u, 0)}{(u + \tilde{\alpha} - i\sigma)}$$

and

$$\varepsilon_{(-)}(\tilde{\alpha}) = 1 + \lim_{\sigma \rightarrow 0_{+}} \int_{-\infty}^{+\infty} du \frac{\bar{D}(u)}{(u + \tilde{\alpha} - i\sigma)}$$

which are defined in a manner equivalent to that given by Eqs. 10 and 11.

We shall next make use of the integral representation

$$\frac{\bar{f}_{1(-)}(k, \tilde{\alpha}, 0)}{\varepsilon_{(-)}(\tilde{\alpha})} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\alpha \frac{1}{\alpha + \tilde{\alpha}} \left[\frac{\bar{f}_{1(+)}(k, \alpha, 0)}{\varepsilon_{(+)}(\alpha)} \right] \quad (19)$$

and substitute Eq. 19 into Eq. 18:

$$\begin{aligned} \bar{f}_1(k, u, t) &= \lim_{\gamma \rightarrow 0_{(+)}} \frac{1}{2\pi i} \int_{-\infty-i\gamma}^{+\infty-i\gamma} d\tilde{\alpha} e^{i\tilde{\alpha}t} \\ &\times \left[\frac{\bar{f}_1(k, u, 0)}{(\tilde{\alpha} + u)} - \bar{D}(u) \int_{-\infty}^{+\infty} d\alpha \frac{1}{(\tilde{\alpha} + u)(\tilde{\alpha} + \alpha)} \frac{\bar{f}_{1(+)}(k, \alpha, 0)}{\varepsilon_{(+)}(\alpha)} \right] \end{aligned} \quad (20)$$

Since

$$\frac{1}{(\tilde{\alpha} + u)(\tilde{\alpha} + \alpha)} = \frac{1}{(\alpha - u)} \left[\frac{1}{\tilde{\alpha} + u} - \frac{1}{\tilde{\alpha} + \alpha} \right]$$

it is evident that one can readily perform the $\tilde{\alpha}$ integration and obtain the following result:

Comparing Eq. 21 with Eq. 16, we conclude that, if $\bar{f}_1(k, u, 0) = \bar{f}_2(k, u, 0)$, then

$$\bar{f}_1(k, u, t) = \bar{f}_1(k, u, 0) e^{-iku t}$$

$$+ \bar{D}(u) \int_{-\infty}^{+\infty} \frac{d\alpha}{(u - \alpha)} (e^{-iku t} - e^{-ika t}) \frac{\bar{f}_{1(+)}(k, \alpha, 0)}{\varepsilon_{(+)}(\alpha)} \quad (21)$$

$$\bar{f}_1(k, u, t) = \bar{f}_2(k, u, t)$$

(Q.E.D.)

IV. CONCLUDING REMARKS

It is well known that, in the analyses by Landau and van Kampen, f^0 is assumed to be Maxwellian, and the dispersion equation does not have roots which should give rise to unstable oscillations. In order to include the

more general cases in the present proof, one may start with Case's solution, rather than van Kampen's solution. Such extension is straightforward and involves no essential difficulty.

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